ADDENDUM TO *HORI-MOLOGICAL PROJECTIVE DUALITY*

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This note is an appendix to [RS16] and we use notation from that paper freely. Throughout this appendix $v = \dim V$ is an even number.

A.1. **A window statement.** Recall that irreps of $\mathrm{GSp}(Q)$ have highest weights (δ, k) for δ a dominant weight of $\mathrm{Sp}(Q)$ and k a weight of Δ such that $\sum_i \delta_i + k \cong 0$ mod 2. We define a subset

$$\Omega \subset Y_{q,s} \times \mathbb{Z}$$

as the set of weights (δ, k) such that either

- $k \in \big[-qv, (q-1)v\big)$, or
- $k \in \big[(q-1)v, qv\big)$ and $\delta \in Y_{q,s-1}$.

Then we define a corresponding full subcategory

$$\mathsf{DB}(\mathcal{Y} \times_{\mathbb{C}^*} L^\perp, W)_\Omega \;\subset\; D^b(\mathcal{Y} \times_{\mathbb{C}^*} L^\perp, W)$$

of objects $\mathcal{E}$ such that $h_\bullet(\mathcal{E}|_0)$ contains only irreps from the set Ω.

Theorem A.1. *The restriction functor*

$$\mathsf{DB}(\mathcal{Y} \times_{\mathbb{C}^*} L^\perp, W)_\Omega \;\longrightarrow\; \mathsf{DB}(\mathcal{Y}^{ss} \times_{\mathbb{C}^*} L^\perp, W)$$

is an equivalence.

This is the analogue of Theorem 4.7 in [RS16] for the case when $v = \dim V$ is even. As in that theorem the real content is the special case $L^\perp = 0$, *i.e.* the equivalence:

$$\mathsf{DB}(\mathcal{Y})_\Omega \xrightarrow{\sim} \mathsf{DB}(\mathcal{Y}^{ss})$$

A.2. **Squeezing Van den Bergh.** Our proof of Theorem A.1 will depend on computations of certain local cohomology groups along the unstable locus $\mathcal{Y} \setminus \mathcal{Y}^{ss}$ in $\mathcal{Y}$. In the paper [VdB91] a partial computation of these groups was carried out, in order to prove Cohen–Macaulayness of certain modules of covariants. As pointed out in that paper the results obtained are not the strongest possible, and we will here refine some statements from the paper to prove what we need.

Van den Bergh's computations are rather complicated and involve several spectral sequences. We will import his techniques and results without explaining them; we apologise that this makes this appendix rather opaque unless the reader is familiar with [VdB91].

A.2.1. *General notation.* We first recall some general notation from [VdB91].

Let G be a reductive group, and let M be a linear representation. Choose a maximal torus T in G, and let $X(T)$ and $Y(T)$ denote the lattices of characters and cocharacters respectively. We write $\Phi_G \subset X(T)$ for the set of roots, and Φ_G^+ for the set of positive roots. We let B be the Borel subgroup defined such that $\Phi_B = -\Phi_G^+$.

Choose a diagonalisation of the action of T on M, and let $\mathrm{Chars}(M)$ be the associated sequence of weights $\alpha_1, \ldots, \alpha_{\dim M} \in X(T)$. Given a cocharacter $\lambda \in Y(T)$, we let $\mathrm{Chars}_\lambda^{\leq 0}(M) \subseteq \{1, \ldots, \dim M\}$ denote the set of i such that $(\alpha_i, \lambda) \leq 0$. We denote the complement of this subset by $\mathrm{Chars}_\lambda^{> 0}(M)$.

Let $\mathcal{Q}$ denote the set of parabolic subgroups containing B, and let:

$$\mathcal{R} = \{(R, P) \in \mathcal{Q} \times \mathcal{Q} \mid R \subseteq P\}$$

Given $(R, P) \in \mathcal{R}$, the notation π_R^P will be used to denote the maps $G/R \to G/P$ and $G \times^R M \to G \times^P M$.

For any cocharacter $\lambda \in Y(T)$, let $M_\lambda = \{x \in M \mid \lim_{t\to 0} \lambda(t) \cdot x = 0\}$. Given a set of cocharacters $U \subset Y(T)$, let $M_U = \bigcup_{\lambda \in U} M_\lambda$. For a parabolic subgroup $P \subseteq G$, let $A_P = \{\lambda \in Y(T) \mid (\lambda, \rho) \geq 0 \text{ for all } \rho \in \Phi_P\}$. We use the abbreviation $M_P = M_{A_P}$.

A.2.2. *Notation in our case.* For the remainder of Section A.2, we let $M = \mathrm{Hom}(V, Q)$ and $G = \mathrm{Sp}(Q)$. We choose the standard maximal torus T and the standard positive roots. Concretely, a character $\chi \in X(T)$ is specified by an q-tuple $(\chi_1, \ldots, \chi_q)$. Letting E_i denote the character with i-th coordinate 1 and all other coordinates 0, the simple positive roots of G are $E_1 - E_2, \ldots, E_{q-1} - E_q$, and $2E_q$. We denote this set by Φ^+_{sim}.

We use the convention that $\mathbf{i}_j$ denotes the sequence where the integer i is repeated j times, so that e.g. $(\mathbf{2}_2, \mathbf{1}_3) = (2, 2, 1, 1, 1)$. For each $i \in [0, q]$ we define a cocharacter $\mu_i = (-\mathbf{1}_i, \mathbf{0}_{q-i}) \in Y(T)$. These will be the only cocharacters that are relevant for our calculations, because of the following:

Lemma A.2. *For $P \in \mathcal{Q}$, we have $M_P = M_{\mu_i}$, where i is the largest integer such that $\mu_i \in A_P$.*

Proof. Suppose $\lambda \in A_P$. Then in particular $\lambda \in A_B$, which means that $(\lambda, \rho) \leq 0$ for all $\rho \in \Phi^+_{\mathrm{sim}}$. Writing $\lambda = (a_1, \ldots, a_n)$, this means $a_1 \leq \cdots \leq a_n \leq 0$. Letting j be the largest integer such that $a_j < 0$, one checks that $M_\lambda = M_{\mu_j}$. Finally, since $M_{\mu_0} \subseteq M_{\mu_1} \subseteq \cdots \subseteq M_{\mu_q}$, and the claim then follows. □

We will also need to consider the group $\overline{G} = \mathrm{GSp}(Q)$, and we write $\overline{T}$ for its maximal torus. A character $\chi \in X(\overline{T})$ is specified by a pair $(\chi_T, \chi_\Delta) = ((\chi_1, \ldots, \chi_q), \chi_\Delta)$, where these are subject to the condition that $\sum_i \chi_i + \chi_\Delta \equiv 0 \pmod 2$.[1] The roots of $\overline{G}$ are $\{(\rho, 0),\ \rho \in \Phi_G\}$.

A.2.3. *Local cohomology computations.* Let $M^{us} \subset M$ denote the locus where the image of V in Q is isotropic; so in the notation of the rest of the paper, we have:

$$\mathcal{Y}^{ss} = \left[(M \setminus M^{us}) \,/\, \mathrm{GSp}(Q)\right]$$

M^{us} is the instability locus for one of the two possible stability conditions for the action of $\overline{G} = \mathrm{GSp}(Q)$, it is also the null-cone for the action of $G = \mathrm{Sp}(Q)$. This means it must be the orbit of the linear subspace M_B, which equals M_{μ_q} by Lemma A.2. One can also see this in an elementary way: the cocharacter μ_q has the effect of scaling a specific Lagrangian subspace in Q, and scaling a complementary Lagrangian in the opposite direction; therefore M_{μ_q} is the locus where V lands inside this specific Lagrangian, and so $M^{us} = GM_{\mu_q}$.

For a character (χ_T, χ_Δ), with χ_T a dominant weight of $\mathrm{Sp}(Q)$, we are interested in computing the $\overline{G}$-invariants in the local cohomology group:

$$H^\bullet_{M^{us}}\left(M,\, \mathbb{S}^{\langle \chi_T, \chi_\Delta \rangle} Q\right)$$

Equivalently, we wish to understand the local cohomology $H^\bullet_{M^{us}}(M, \mathcal{O}_M)$ as a representation of $\overline{G}$.

[1] In previous sections we have denoted this data by (δ, k).

Remark A.3. The computations in [VdB91] only concern a single group G, and they compute local cohomology along the null-cone for G, as a G-representation. However, the difference between G-representations and $\overline{G}$-representations is just an additional grading, and all the computations remain valid if we keep track of this too. Note that we can't just replace G by $\overline{G}$ in Van den Bergh's method, because the null-cone for the $\overline{G}$-action is the whole of M.

Lemma A.4. *Let $\lambda \in Y(T)$. A $\overline{T}$-character χ appears in $H^*_{M_\lambda}(M, \mathcal{O}_M)$ with multiplicity equal to the number of sequences $(a_i), (b_i)$ such that*

$$\chi = \sum_{i \in \mathrm{Chars}_\lambda^{\leq 0}(M)} (a_i + 1)\alpha_i - \sum_{j \in \mathrm{Chars}_\lambda^{>0}(M)} b_j \alpha_j \qquad a_i, b_i \in \mathbb{Z}_{\geq 0}.$$

Proof. This follows from [VdB93, Prop. 3.3.1]. Note that λ is a cocharacter for $T \subset G$, and χ is a character for $\overline{T} \subset \overline{G}$; see Remark A.3. □

Lemma A.5.

(1) *The $\overline{T}$-characters appearing in $H^*_{M_{\mu_0}}(M, \mathcal{O}_M) = H^*_0(M, \mathcal{O}_M)$ are of the form*
$$((d_1, \ldots, d_q), 2qv + k) \qquad d_i, k \in \mathbb{Z}$$
with $k \geq \sum |d_i|$.

(2) *For $i = 1, \ldots, s$, the $\overline{T}$-characters appearing in $H^*_{M_{\mu_i}}(M, \mathcal{O}_M)$ are of the form*
$$((v + c_1, \ldots, v + c_i, d_{i+1}, \ldots, d_q), 2qv - iv + k) \qquad c_i, d_i, k \in \mathbb{Z}$$
with $c_j \geq 0$, and with $\sum c_j + |k| \geq \sum |d_j|$ and $k + \sum c_i \geq 0$.
The character $((\mathbf{v}_i, \mathbf{0}_{q-i}), 2qv - iv)$ appears exactly once.

Proof. Straightforward from Lemma A.4. □

We now have enough information to prove the vanishing of certain local cohomology groups.

Proposition A.6. *Let $\chi = (\chi_T, \chi_\Delta) \in X(\overline{T})^+$. Assume that one of the following holds:*

(1) *$\chi_T \in Y_{q,2s-1}$ and $\chi_\Delta < 2qv$.*
(2) *$\chi_T \in Y_{q,2s}$ and $\chi_\Delta < 2qv - v$.*

Then $\mathbb{S}^{\langle \chi_T, \chi_\Delta \rangle} Q$ does not appear as a summand of $H^*_{M^{\mathrm{us}}}(M, \mathcal{O}_M)$, or equivalently:

$$H^*_{M^{\mathrm{us}}}\left(M, \mathbb{S}^{\langle \chi_T, -\chi_\Delta \rangle} Q\right)^{\overline{G}} = 0$$

Proof. Corollary 6.8 of [VdB91] says that (bearing in mind Remark A.3) if $\mathbb{S}^{\langle \chi_T, \chi_\Delta \rangle} Q$ appears in $H^*_{M^{\mathrm{us}}}(M, \mathcal{O}_M)$ we must have

$$\chi = \chi' + \sum_{\rho \in S} \rho \tag{A.7}$$

where χ' is a $\overline{T}$-character appearing in some $H^*_{M_\lambda}(M, \mathcal{O}_M)$, and S is some subset of $\Phi_{\overline{G}}$.

By the proof, we see that in fact we must have χ' appearing in some $H^*_{M_P}(M, \mathcal{O}_M)$ for some $P \in \mathcal{Q}$, hence χ' appears in some $H^*_{M_{\mu_i}}(M, \mathcal{O}_M)$, by Lemma A.2.

We define the following two norms on the space $X(T)_\mathbb{Q}$:

$$|\psi|_1 = \max_i |\psi_i| \qquad \text{and} \qquad |\psi|_2 = \max_{i \neq j} \frac{|\psi_i| + |\psi_j|}{2}$$

We'll use the same notation for the pull-back of these functions to $X(\overline{T})_\mathbb{Q}$; there they become only semi-norms, but still satisfy the triangle inequality.

Suppose that condition (1) holds. Then in (A.7) we must have $|\sum_{\rho\in S}\rho|_1 \leq 2q$, and by assumption $|\chi|_1 \leq 2s-1$, hence:

$$|\chi'|_1 \leq 2s+2q-1 = v-1$$

Using Lemma A.5 and the fact that $\chi_\Delta = \chi'_\Delta < 2qv$, we see that χ cannot appear in $H^*_{M_{\mu_i}}(M, \mathcal{O}_M)$ for any i, and hence $\mathbb{S}^{\langle \chi_T, \chi_\Delta\rangle}Q$ does not appear in $H^*_{M^{\mathrm{us}}}(M, \mathcal{O}_M)$.

Suppose next that condition (2) holds. In (A.7) we must have $|\sum_{\rho\in S}\rho|_2 \leq 2q-1$ and by assumption $|\chi|_2 \leq 2s$, hence $|\chi|_2 \leq v-1$. By Lemma A.5 and the fact that $\chi_\Delta = \chi'_\Delta < 2qv - v$, we see that if χ appears in $H^*_{M_{\mu_i}}(M, \mathcal{O}_M)$, we must have $i=1$. But in the terminology of that lemma, we must have $k<0$, and hence $c_1>0$, which implies $|\chi'|_1 > v$, and hence $|\chi|_1 > 2s$, which contradicts our assumed condition on χ. Hence $\mathbb{S}^{\langle \chi_T, \chi_\Delta\rangle}Q$ does not appear in $H^*_{M^{\mathrm{us}}}(M, \mathcal{O}_M)$. □

We next want to prove that certain local cohomology groups do *not* vanish. This is harder, and we have to look deeper into Van den Bergh's method.

We present the essential part of the argument first; we then follow this with several computational lemmas that it requires.

Proposition A.8. *Let $\delta_T = (2s, \mathbf{0}_{q-1})$ and $\delta_\Delta = 2qv - v$. The irrep $\mathbb{S}^{\langle \delta_T, \delta_\Delta\rangle}Q$ appears in $H^*_{M^{\mathrm{us}}}(M, \mathcal{O}_M)$, and so:*

$$H^*_{M^{\mathrm{us}}}\left(M,\, \mathbb{S}^{\langle \delta_T, -\delta_\Delta\rangle}Q\right)^{\overline{G}} \neq 0$$

Proof. We will show that $\mathbb{S}^{\langle \delta_T, \delta_\Delta\rangle}Q$ appears in $H^*_{M^{\mathrm{us}}}(M, \mathcal{O}_M)$ with odd multiplicity. The strategy is to compute $H^*_{M^{\mathrm{us}}}(M, \mathcal{O}_M)$ via several spectral sequences from [VdB91], and apply the observation that working modulo 2, the multiplicity of an irrep in any page of a spectral sequence equals its multiplicity in the E^∞-page. The bigradings on the spectral sequences involved are irrelevant and will not be specified.

By [VdB91, Thm. 5.2.1], there is a spectral sequence converging to $H^*_{M^{\mathrm{us}}}(M, \mathcal{O}_M)$ whose E^1-page is

$$\bigoplus_{(R,P)\in\mathcal{R}} H^{DR}_*(G\times^R M_P/M).$$

Here H^{DR}_* denotes relative algebraic de Rham homology, see [VdB91].

Each summand $H^{DR}_*(G\times^R M_P/M)$ is computed by a spectral sequence whose E^1-page is

$$H^*_{G\times^B M_P}\big(G\times^B M,\, (\pi^B_R)^*(\wedge^\bullet \Omega_{G\times^R M/M})\big), \tag{A.9}$$

by the proof of [VdB91, Lemmas 6.2 & 6.3]. There is a spectral sequence converging to (A.9), whose E^2-page is

$$H^*\big(G/B,\, (\pi^B_R)^*(\wedge^\bullet \Omega_{G/R}) \otimes \widetilde{H^*_{M_P}(M, \mathcal{O}_M)}\big), \tag{A.10}$$

by [VdB91, Lemma 6.4]. Here the notation $\widetilde{H^*_{M_P}(M, \mathcal{O}_M)}$ denotes the vector bundle on G/B induced by treating $H^*_{M_P}(M, \mathcal{O}_M)$ as a B-representation.

Let $n(R,P)\in \mathbb{Z}/2\mathbb{Z}$ be the multiplicity of $\mathbb{S}^{\langle \delta_T, \delta_\Delta\rangle}Q$ in $H^{DR}_*(G\times^R M_P/M)$ (taken modulo 2), this is the same as its multiplicity in (A.10). The (mod 2)-multiplicity of $\mathbb{S}^{\langle \delta_T, \delta_\Delta\rangle}Q$ in $H^*_{M^{\mathrm{us}}}(M, \mathcal{O}_M)$ is $\sum_{(R,P)\in\mathcal{R}} n(R,P)$, and by Lemma A.15 below we find that this equals 1. □

Let $P_1 \in \mathcal{Q}$ be the following parabolic in G:

$$P_1 = \{g\in G \mid \lim_{t\to 0} \mu_1(t) g \mu_1(t)^{-1} \text{ exists}\}$$

This is the parabolic subgroup whose roots are $\Phi_B \cup \{\alpha\in\Phi_G \mid (\alpha,\mu_1)\geq 0\}$.

Lemma A.11. *Choose a parabolic $P \in \mathcal{Q}$. Then the character $((v, \mathbf{0}_{q-1}), 2qv - v)$ appears in $H^*_{M_P}(M, \mathcal{O}_M)$ with multiplicity 1 if $P = P_1$, and does not appear if $P \neq P_1$.*

Proof. Let $S = \Phi^+_{\text{sim}} \cap \Phi_P$. If $(1, -1, \mathbf{0}_{q-2}) \in S$, then $\mu_1 \notin A_P$. If there is an $i \in [2, q-1]$ such that $(\mathbf{0}_{i-1}, 1, -1, \mathbf{0}_{q-i-1}) \notin S$, then $\mu_i \in A_P$. If $(\mathbf{0}_{q-1}, 2) \notin S$, then $\mu_q \in A_P$.

Thus we have "$\mu_1 \in A_P$ and $\mu_i \notin A_P$ for all $i > 1$" if and only if $S = \Phi^+_{\text{sim}} \setminus (1, -1, \mathbf{0}_{q-2})$, which is if and only if $P = P_1$. Combining Lemmas A.5 and A.2 we get the claim we want. □

For $\chi \in \overline{T}$, let $BWB(\chi) = w(\chi + \overline{\rho}) - \overline{\rho}$ if there exists a (necessarily unique) Weyl group element $w \in W_G$ such that $w(\chi + \overline{\rho}) - \overline{\rho}$ is a dominant weight; if no such w exists, let $BWB(\chi)$ be undefined. (The notation BWB here is short for the Borel–Weil–Bott theorem, which is how this operation on characters enters the computation.)

Lemma A.12. *Let C be the set of pairs (R, S), with $R \in \mathcal{Q}$, and $S \subset -\Phi_{G/P}$ such that $BWB(\sum_{\rho \in S} \rho) = 0$. Then $|C|$ is odd.*

Proof. If (R, S) is such a pair, then there exists a $w \in W_G$ such that $w(\sum_{\rho \in S} \rho + \overline{\rho}) - \overline{\rho} = 0$. Since $S \subseteq -\Phi^+$, this tranforms into

$$w\Big(\sum_{\rho \in S} \rho + \sum_{\rho \in \Phi^+ \setminus -S} \rho\Big) = \sum_{\rho \in \Phi^+} \rho.$$

Since the left hand side is a sum of distinct roots, and the right hand side admits only one expression as a sum of distinct roots, we have $S \cup (\Phi^+ \setminus -S) = w^{-1}(\Phi^+)$. From this it follows that

$$S = w^{-1}(\Phi^+) \cap \Phi^-. \tag{A.13}$$

We next note that

$$S \subseteq -\Phi_{G/P} \;\Leftrightarrow\; -\Phi_P \cap S = \varnothing \;\Leftrightarrow\; -\Phi_P \cap -\Phi^+_{\text{sim}} \cap S = \varnothing, \tag{A.14}$$

where $\Phi^+_{\text{sim}} \subseteq \Phi^+$ are the simple roots. To see the last equivalence, note that $-\Phi_P \cap S \subseteq -\Phi_P \cap -\Phi^+$, and any element in $-\Phi_P \cap -\Phi^+$ can be written as a sum of elements in $-\Phi_P \cap -\Phi^+_{\text{sim}}$ [Hum75, p. 184]. If

$$\varnothing = -\Phi_P \cap -\Phi^+_{\text{sim}} \cap S = -\Phi_P \cap -\Phi^+_{\text{sim}} \cap w^{-1}(\Phi^+) \cap \Phi^- = -\Phi_P \cap w^{-1}(\Phi^+),$$

then

$$-\Phi_P \cap -\Phi^+_{\text{sim}} \subseteq \Phi \setminus w^{-1}(\Phi^+) = w^{-1}(-\Phi^+).$$

Since $w^{-1}(-\Phi^+)$ is closed under addition, it follows that $-\Phi_P \subseteq w^{-1}(-\Phi^+)$ and then since $w^{-1}(-\Phi^+) \cap S = \varnothing$, we get $-\Phi_P \cap S = \varnothing$.

The map $P \mapsto -\Phi_P \cap -\Phi^+_{\text{sim}}$ gives a bijection between $\mathcal{Q}$ and the power set of $\{-\Phi^+_{\text{sim}}\}$. Thus by (A.14), we find that for a fixed S, the number of P such that $S \subseteq -\Phi_{G/P}$ equals $2^{|-\Phi^+_{\text{sim}} \setminus S|}$. Hence the parity of $|C|$ equals the number of S such that $-\Phi^+_{\text{sim}} \subseteq S$. But $-\Phi^+_{\text{sim}} \subseteq S$ implies that in (A.13) we must take $w \in W_G$ to be the element which acts via multiplication by -1 on $X(T)$. In particular there is only one such S, hence we are done. □

Let $\delta = (\delta_T, \delta_\Delta) = ((2s, \mathbf{0}_{q-1}), 2qv - v)$ as above.

Lemma A.15. *Let $P \in \mathcal{Q}$. The representation $\mathbb{S}^{\langle \delta_T, \delta_\Delta \rangle} Q$ appears in*

$$\bigoplus_{(R,P) \in \mathcal{R}} H^*\big(G/B,\, (\pi^B_R)^*(\wedge^\bullet \Omega_{G/R}) \otimes_{\mathcal{O}_{G/B}} H^*_{M_P}\widetilde{(M, \mathcal{O}_M)}\big) \tag{A.16}$$

with odd multiplicity if $P = P_1$, and does not appear if $P \neq P_1$.

Proof. In the following, we identify $\Phi_{\overline{G}}$ with Φ_G in the obvious way, so that a $\rho \in \Phi_G$ is thought of as an element of $X(\overline{T})$.

Let us first consider one summand of (A.16), i.e. one corresponding to a fixed $(R, P) \in \mathcal{R}$. By the proof of [VdB91, Lemma 6.5], the multiplicity of the representation $\mathbb{S}^{\langle \delta_T, \delta_\Delta \rangle} Q$ in this summand equals the cardinality of the set

$$\Big\{(\chi, S) \;\mid\; \chi \text{ appears in } H^*_{M_P}(M, \mathcal{O}_M),\; S \subseteq -\Phi_{G/R},\; BWB(\chi + \sum_{\rho \in S} \rho) = \delta\Big\} \tag{A.17}$$

where (χ, S) is counted with the multiplicity of χ in $H^*_{M_P}(M, \mathcal{O}_M)$. Since the diagonal cocharacter $\Delta \in Y(\overline{T})$ is fixed by $W_{\overline{G}}$, and $\rho_\Delta = 0$ for all $\rho \in \Phi(G/R)$, we see that any χ appearing in (A.17) must satisfy $\chi_\Delta = 2qv - v$.

Let $\psi = ((v, \mathbf{0}_{q-1}), 2qv - v) \in X(\overline{T})$. We now claim that if (χ, S) appears in (A.17), then $\chi = \psi$ and $P = P_1$. So suppose that (χ, S) appears, and let $\chi' = \chi + \sum_{\rho \in S} \rho$. If $BWB(\chi') = \delta$, then we must have $\chi'_1 \leq 2s$ and $\chi'_2 \leq 2s + 1$. In general, we have $(\sum_{\rho \in S} \rho)_1 \geq -2q$ and $(\sum_{\rho \in S} \rho)_2 \geq -2q + 2$. It follows that $\chi_1 \leq v$ and $\chi_2 \leq v - 1$.

We know that χ appears in $H^*_{M_P}(M, \mathcal{O}_M) = H^*_{M_{\mu_i}}(M, \mathcal{O}_M)$ for some $i \in [0, q]$. The inequality $\chi_2 \leq v - 1$ implies that $i \leq 1$, while $\chi_\Delta = 2qv - v$ implies $i \neq 0$, hence $i = 1$, and so we must have $P = P_1$, by Lemma A.11.

Finally, since $\chi_1 \leq v$, Lemma A.5 shows that $\chi = \psi$, and that χ_1 appears precisely once in $H^*_{M_P}(M, \mathcal{O}_M)$.

Hence the cardinality of (A.17) equals the cardinality of

$$\Big\{S \subseteq -\Phi_{G/R} \;\mid\; BWB(\sum_{\rho \in S} \rho + \chi) = \delta\Big\} \tag{A.18}$$

if $P = P_1$, and equals 0 otherwise.

Let $A = \{-E_1 \pm E_i\}_{i \in [2,q]} \cup \{-2E_1\} \subseteq -\Phi^+$. Since $(\sum_{\rho \in S} \rho + \psi)_1 \leq 2s$, we must have $(\sum_{\rho \in S} \rho)_1 \leq -2q$, which implies that $(\sum_{\rho \in S} \rho)_1 = -2q$ and hence $A \subseteq S$. Let now $S' = S \setminus A$. The set (A.18) is then in bijection with the set

$$\Big\{S' \subseteq -\Phi_{\overline{G}/R} \setminus A \;\mid\; BWB\Big(\sum_{\rho \in S'} \rho + ((2q, \mathbf{0}_{q-1}), 0)\Big) = \delta\Big\}.$$

Let now Ψ be the root system for $\mathrm{GSp}(2q-2)$, and let E'_i be the standard generators of the character lattice of $\mathrm{Sp}(2q-2)$. There is an inclusion of the character lattice of $\mathrm{GSp}(2q-2)$ into that of $\mathrm{GSp}(2q)$ by

$$(E'_i, k) \mapsto (E_{i+1}, k).$$

This gives a bijection $F : -\Phi^+ \setminus A \to -\Psi^+$. The map F then also induces a bijection between the $R \in \mathcal{Q}$ such that $R \subseteq P_1$ and the set of all parabolic R in $\mathrm{GSp}(2q-2)$ containing the standard Borel subgroup.

Furthermore, it is easy to see that $BWB\big(\sum_{\rho \in S'} \rho + ((2q, \mathbf{0}_{q-1}), 0)\big) = \delta$ holds if and only if $BWB(\sum_{\rho \in F(S')} \rho) = 0$, where the latter equation is in the character lattice of $\mathrm{GSp}(2q-2)$. Thus Lemma A.12 (applied to $G = \mathrm{Sp}(2q-2)$) concludes the proof. □

This completes the proof of Proposition A.8. We need to prove one more non-vanishing result, as follows:

Proposition A.19. *Let $\chi = (\chi_T, \chi_\Delta) = ((\mathbf{0}_q), 2qv)$. Then $\mathbb{S}^{\langle \chi_T, \chi_\Delta \rangle} Q$ appears as a summand of $H^*_{M^{\mathrm{us}}}(M, \mathcal{O}_M)$, so:*

$$H^*_{M^{\mathrm{us}}}\Big(M, \mathbb{S}^{\langle \chi_T, -\chi_\Delta \rangle} Q\Big)^{\overline{G}} \neq 0$$

Proof. An argument similar to that in the proof of Lemma A.15 shows that if $\mathbb{S}^{\langle \chi_T, \chi_\Delta\rangle} Q$ appears in

$$H^*\big(G/B,\, (\pi_R^B)^*(\wedge^\bullet \Omega_{G/R}) \otimes_{\mathcal{O}_{G/B}} H^*_{M_P}\widetilde{(M, \mathcal{O}_M)}\big)$$

then we must have $M_P = M_{\mu_0} = 0$, and arguing similarly to Lemma A.11, this means that $P = G$. The proof of Lemma A.15 shows that the multiplicity of $\mathbb{S}^{\langle \chi_T, \chi_\Delta\rangle} Q$ inside

$$\bigoplus_{R \in \mathcal{Q}} H^*\big(G/B,\, (\pi_R^B)^*(\wedge^\bullet \Omega_{G/R}) \otimes_{\mathcal{O}_{G/B}} H^*_0\widetilde{(M, \mathcal{O}_M)}\big)$$

reduces to the cardinality of the pairs (R, S) such that $R \in \mathcal{Q}$, $S \in -\Phi_{G/R}$, and $BWB(\sum_{\rho \in S} \rho) = 0$. But by Lemma A.12 there is an odd number of such. Arguing as in the proof of Proposition A.8 completes the proof. □

A.3. **Proof of the window equivalence in the even case.** We will now use the results from the previous section to prove Theorem A.1, essentially in the same way as we proved [RS16, Theorem 4.7] when v was odd. We will prove the essential surjectivity and fully faithfulness of the functor separately.

A.3.1. *Fully faithfulness.*

Lemma A.20. *Given $\chi, \psi \in X(T)^+$, every summand $\mathbb{S}^{\langle\alpha\rangle} Q$ of $\mathbb{S}^{\langle\chi\rangle} Q \otimes \mathbb{S}^{\langle\psi\rangle} Q$ satisfies $\alpha_1 \leq \chi_1 + \psi_1$.*

Proof. Given any character $\phi \in X(T)$, let $|\phi|_1 = \max |\phi_i|$. Let now ϕ be a T-weight occurring in the $\mathrm{Sp}(Q)$-irrep $\mathbb{S}^{\langle\delta\rangle} Q$. We claim

$$|\phi|_1 \leq |\delta|_1. \tag{A.21}$$

Suppose not, then there would be a k such that $|\phi_k| > |\delta|_1 = \delta_1$. We can find an element w of the Weyl group of $\mathrm{Sp}(Q)$ such that $(w\phi)_1 = |\phi_k|$. But since $w\phi$ is a T-weight of $\mathbb{S}^{\langle\delta\rangle} Q$, it is dominated by δ, hence $(w\phi)_1 \leq \delta_1$, and we have a contradiction, so (A.21) holds.

Now $\alpha = \phi + \phi'$ for ϕ some T-weight of $\mathbb{S}^{\langle\chi\rangle} Q$ and ϕ' some T-weight of $\mathbb{S}^{\langle\psi\rangle} Q$, so we find:

$$\alpha_1 = |\alpha|_1 \;\leq\; |\phi|_1 + |\phi'|_1 \;\leq\; |\chi|_1 + |\psi|_1 = \chi_1 + \phi_1$$

□

Proposition A.22. *Let $\chi, \psi \in \Omega$. Then restriction induces an isomorphism:*

$$\mathrm{Hom}_{\mathcal{Y}}\big(\mathbb{S}^{\langle \chi_T, \chi_\Delta\rangle} Q,\, \mathbb{S}^{\langle \psi_T, \psi_\Delta\rangle} Q\big) \;\xrightarrow{\sim}\; \mathrm{Hom}_{\mathcal{Y}^{ss}}\big(\mathbb{S}^{\langle \chi_T, \chi_\Delta\rangle} Q,\, \mathbb{S}^{\langle \psi_T, \psi_\Delta\rangle} Q\big)$$

Proof. We need to prove the vanishing of the local cohomology of the bundle $\mathbb{S}^{\langle \chi_T, \chi_\Delta\rangle} Q^\vee \otimes \mathbb{S}^{\langle \psi_T, \psi_\Delta\rangle} Q$ along the locus $\mathcal{Y} \setminus \mathcal{Y}^{ss}$. Examining the definition of Ω, it's sufficient to check the following two cases:

a) $\chi_T \in Y_{q,s-1},\ \psi_T \in Y_{q,s},\ \chi_\Delta - \psi_\Delta < 2qv$.
b) $\chi_T \in Y_{q,s},\ \psi_T \in Y_{q,s},\ \chi_\Delta - \psi_\Delta < 2qv - v$.

Let $\mathbb{S}^{\langle \alpha_T, \alpha_\Delta\rangle} Q$ be a summand of $\mathbb{S}^{\langle \chi_T, \chi_\Delta\rangle} Q^\vee \otimes \mathbb{S}^{\langle \psi_T, \psi_\Delta\rangle} Q$. In case (a) we must have $\alpha_T \in Y_{q,2s-1}$ (by Lemma A.20) and $\alpha_\Delta = \psi_\Delta - \chi_\Delta < 2qv$; this is the first hypothesis of Lemma A.6 so the result follows. Similarly in case (b) we have $\alpha_T \in Y_{q,2s}$ and the second hypothesis of that lemma is satisfied. □

Obviously this result still holds if we work on $\mathcal{Y} \times_{\mathbb{C}^*} L^\perp$ instead, and then the first paragraph of the proof of Proposition 4.13 in [RS16] shows immediately that:

Corollary A.23. *The restriction functor*

$$\mathsf{DB}(\mathcal{Y} \times_{\mathbb{C}^*} L^\perp,\, W)_\Omega \to \mathsf{DB}(\mathcal{Y}^{\mathrm{ss}} \times_{\mathbb{C}^*} L^\perp,\, W)$$

is fully faithful.

A.3.2. *Essential surjectivity.* As in the v odd case, we define an object

$$P_{\delta,k} \in \mathsf{DB}(\mathcal{Y})$$

for any dominant weight (δ, k) of $\mathrm{GSp}(Q)$, by projecting the twist $\mathcal{O}_0 \otimes \mathbb{S}^{\langle \delta,k\rangle}Q$ of the skyscraper sheaf at the origin into the admissible subcategory $\mathsf{DB}(\mathcal{Y}) \subset D^b(\mathcal{Y})$. As before these objects restrict to zero in $\mathsf{DB}(\mathcal{Y}^{ss})$ [RS16, Lemma 4.16] and each one has a finite resolution by vector bundles which hence becomes an exact sequence on $\mathcal{Y}^{ss}$. The bundles appearing in this sequence correspond to the irreps in $h_\bullet(P_{\delta,k}|_0)$, and as before we refer to these as the *weights* of $P_{\delta,k}$.

Proposition A.24. *Let (δ, k) be a dominant weight of $\mathrm{GSp}(Q)$. Then the set of weights of $P_{\delta,k}$ contains one copy of (δ, k). Furthermore:*

- *If $\delta \in Y_{q,s-1}$, then the remaining weights are contained in*

$$Y_{s,q} \times [k - 2qv,\, k).$$

- *If $\delta \in Y_{q,s} \setminus Y_{q,s-1}$ then the remaining weights are contained in*

$$Y_{s,q} \times [k - 2(q-1)v,\, k).$$

Proof. The first claim, and the fact that all remaining weights lie in $Y_{s,q} \times (-\infty, k)$, can be argued exactly as in the v odd case (see the proof of [RS16, Lemma 4.15]); we just have to prove the lower bounds.

So, let m be the minimum of the set $\{m' \mid (\psi, m') \text{ is a weight of } P_{\delta,k}\}$. If we consider the locally-free resolution of $P_{\delta,k}$, we get a short exact sequence:

$$(P_{\delta,k})_{>min} \to P_{\delta,k} \to (P_{\delta,k})_{min} \tag{A.25}$$

Here $(P_{\delta,k})_{min}$ is the final term in the resolution, it is a direct sum of vector bundles of the form $\mathbb{S}^{\langle \psi,m\rangle}Q$. The object $(P_{\delta,k})_{>min}$ consists of all the remaining terms, it is a complex of vector bundles whose summands are all of the form $\mathbb{S}^{\langle \psi,m'\rangle}Q$ with $m < m' \leq k$.

Now choose one of the minimal weights (ψ, m), *i.e.* choose one of the summands of $(P_{\delta,k})_{min}$. We make the following claim, which immediately implies the proposition:

- If $\psi \in Y_{q,s-1}$ then $\psi = \delta$ and $m = k - 2qv$.
- If $\psi \in Y_{q,s} \setminus Y_{q,s-1}$ then $\psi = \delta$ and $m = k - 2(q-1)v$.

(So in particular $(P_{\delta,k})_{min}$ is actually a direct sum of copies of $\mathbb{S}^{\langle \delta,m\rangle}Q$.)

For any vector bundle E on $\mathcal{Y}$, let's write $\mathrm{Hom}_{\mathcal{Y}^{us}}(E, -)$ for the functor that sends an object $\mathcal{F}$ to the local cohomology of $\mathrm{Hom}(E, \mathcal{F})$ along the substack $\mathcal{Y}^{us} = [M^{us}/\mathrm{GSp}(Q)]$. To prove our claim we're going to apply this functor to the short-exact-sequence (A.25); we get the two statements by using different choices of vector bundle E.

Let's start with the case $\psi \in Y_{q,s-1}$. Set E to be the bundle $\mathbb{S}^{\langle \psi,m+2qv\rangle}Q$, and apply the functor. To evaluate the first term, we need to know the local cohomology of each bundle

$$\mathrm{Hom}\big(\mathbb{S}^{\langle \psi,m+2qv\rangle}Q,\, \mathbb{S}^{\langle \psi',m'\rangle}Q\big)$$

with $m' > m$. But this always vanishes, by part (1) of Lemma A.6 (c.f. the proof of Proposition A.22). So

$$\mathrm{Hom}_{\mathcal{Y}^{us}}\big(\mathbb{S}^{\langle \psi,m+2qv\rangle}Q,\, (P_{\delta,k})_{>min}\big) = 0$$

and we conclude that the map

$$\mathrm{Hom}_{\mathcal{Y}^{us}}\big(\mathbb{S}^{\langle \psi,m+2qv\rangle}Q,\, P_{\delta,k}\big) \longrightarrow \mathrm{Hom}_{\mathcal{Y}^{us}}\big(\mathbb{S}^{\langle \psi,m+2qv\rangle}Q,\, (P_{\delta,k})_{min}\big) \tag{A.26}$$

is an isomorphism.

Let's examine the third term, the target of this isomorphism (A.26). The bundle $(P_{\delta,k})_{min}$ contains $\mathbb{S}^{\langle \psi, m\rangle}Q$ as a summand, and the bundle

$$\mathrm{Hom}\big(\mathbb{S}^{\langle \psi, m+2qv\rangle}Q,\, \mathbb{S}^{\langle \psi, m\rangle}Q\big)$$

contains $\mathbb{S}^{\langle (\mathbf{0}_q), -2qv\rangle}Q$ as a summand. So by Proposition A.19, we have:

$$\mathrm{Hom}_{\mathcal{Y}^{us}}\big(\mathbb{S}^{\langle \psi, m+2qv\rangle}Q,\, (P_{\delta,k})_{min}\big) \neq 0$$

This means that the second term (the source in (A.26)) is also non-zero. However, the object $P_{\delta,k}$ is supported along $\mathcal{Y}^{us}$, so for this term the local cohomology is just the global sections of the Hom bundle. But by [RS16, Lemma 4.14] this can only be non-zero if we have $\delta = \psi$ and $k = m + 2qv$. This proves the first case of the claim.

Now we consider the second case, when $\psi \in Y_{q,s} \setminus Y_{q,s-1}$. We set E to be the bundle $\mathbb{S}^{\langle \psi, m+2(q-1)v\rangle}Q$, and proceed in a very similar way. The vanishing of the first term now holds by part (2) of Lemma A.6. To understand the third term we need the observation that since the first entry of ψ is s, the bundle

$$\mathrm{Hom}\big(\mathbb{S}^{\langle \psi, m+2(q-1)v\rangle}Q,\, \mathbb{S}^{\langle \psi, m\rangle}Q\big)$$

contains $\mathbb{S}^{\langle (2s, \mathbf{0}_{q-1}), -2(q-1)v\rangle}Q$ as a summand; see Lemma A.29 below. Then Proposition A.8 shows that the third term does not vanish. The remainder of the argument is identical to the first case. □

A trivial modification of the proof of [RS16, Lemma 4.17] yields:

Corollary A.27. *For any $(\chi_T, \chi_\Delta) \in Y_{q,s} \times \mathbb{Z}$, the vector bundle $\mathbb{S}^{\langle \chi_T, \chi_\Delta\rangle}Q$ on $\mathcal{Y}^{ss}$ has a finite resolution by vector bundles associated to the set Ω.*

Now we can use the proof of [RS16, Proposition 4.18] verbatim to deduce that:

Corollary A.28. *The restriction functor*

$$\mathsf{DB}(\mathcal{Y} \times_{\mathbb{C}^*} L^\perp,\, W)_\Omega \to \mathsf{DB}(\mathcal{Y}^{\mathrm{ss}} \times_{\mathbb{C}^*} L^\perp,\, W)$$

is essentially surjective.

This completes the proof of Theorem A.1, except for the following representation-theoretic calculation needed in the proof of Proposition A.24.

Lemma A.29. *Let χ be a dominant weight of $\mathrm{Sp}(Q)$. Then $(\mathbb{S}^{\langle\chi\rangle}Q)^{\otimes 2}$ contains $\mathbb{S}^{\langle (2\chi_1, \mathbf{0}_{q-1})\rangle}Q$ as a summand.*

Proof. The claim is equivalent to the claim that $\mathbb{S}^{\langle (2\chi_1, \mathbf{0}_{q-1})\rangle}Q \otimes \mathbb{S}^{\langle\chi\rangle}Q$ contains $\mathbb{S}^{\langle\chi\rangle}Q$ as a summand. We apply the Littlewood–Newell formula [BKW83, 5.7]. The formula tells us that if β is a partition and n_β denotes the multiplicity of $\mathbb{S}^{\langle\beta\rangle}Q$ in $\mathbb{S}^{\langle\alpha\rangle}Q \otimes \mathbb{S}^{\langle\psi\rangle}Q$, we have the equality

$$\sum n_\beta x^\beta = \sum_{\lambda,\gamma,\theta} c(\beta) N^{\psi}_{\lambda,\gamma} N^{\alpha}_{\lambda,\theta} N^{\beta}_{\gamma,\theta} x^{m(\beta)}.$$

Here the $N^{*}_{*,*}$ are Littlewood–Richardson coefficients, and we sum over all partitions. We have $c(\beta) \in \{-1, 0, 1\}$ and $m(\beta)$ is a partition; both are determined by 'modification rules' [BKW83, Sec. 3]. We only need to know that if the length of β is at most q then $c(\beta) = 1$ and $m(\beta) = \beta$, and if the length of β is $q+1$ then $c(\beta)$ is 0 (and $m(\beta)$ is undefined).

Let us write (i) for the partition $(i, \mathbf{0}_\infty)$. For the product $\mathbb{S}^{\langle\chi\rangle}Q \otimes \mathbb{S}^{\langle (2\chi_1)\rangle}Q$, the Littlewood–Newell formula gives:

$$\sum_\beta n_\beta x^\beta = \sum_{\theta,\beta} \sum_{k=0}^{2\chi_1} c(\beta) N^{\chi}_{(k),\theta} N^{\beta}_{(2\chi_1 - k),\theta} x^{m(\beta)},$$

since $N^{(2\chi_1)}_{\lambda,\gamma} = 1$ if $\lambda = (k)$ and $\gamma = (2\chi_1 - k)$ and $N^{(2\chi_1)}_{\lambda,\gamma} = 0$ otherwise. To find the coefficient n_χ here we must find all β which appear with non-zero coefficient in this sum and which satisfy $m(\beta) = \chi$. However, if $N^{\chi}_{(k),\theta} \neq 0$, then $l(\theta) \leq l(\chi)$, which if $N^{\beta}_{(2\chi_1-k),\theta} \neq 0$ means:

$$l(\beta) \leq l(\theta) + 1 \leq l(\chi) + 1 \leq q + 1$$

So by the modification rules, the only contribution to n_χ comes from $\beta = \chi$. For both $N^{\chi}_{(k),\theta}$ and $N^{\chi}_{(2\chi_1-k),\theta}$ to be non-zero we must have $k = \chi_1$, so:

$$n_\chi = \sum_\theta (N^{\chi}_{(\chi_1),\theta})^2$$

Since $N^{\chi}_{(\chi_1),\theta} = 1$ if $\theta = (\chi_2, \chi_3, \ldots, \chi_q)$ and is zero otherwise, this shows that $n_\chi = 1$. □

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